The Schwarzschild Horizon as a Stochastic Interface: The Emergence of Edwards-Wilkinson Universality

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Abstract

We derive the stochastic partial differential equation (PDE) governing shape fluctuations of the horizon of an evaporating Schwarzschild black hole, working within the slowly evolving trapping horizon formalism of Booth and Fairhurst and semiclassical gravity. The horizon is parametrized as $r = r_0 + \epsilon h(v, \Omega)$ with $r_0 = 2M$ in Eddington-Finkelstein coordinates, and $\epsilon \ll 1$ the slowly evolving parameter. Expanding the marginally trapped surface condition $\theta_{(l)} = 0$ to $O(\epsilon^3)$ in Booth-Fairhurst gauge, we obtain

$$\partial_v h = \frac{l(l+1)-1}{4M} h - \frac{\epsilon^2}{2M} h(\nabla h)^2 + \eta,$$

where ∇ denotes the covariant gradient on the unit S^2 and η is Hawking noise. The leading nonlinearity is $h(\nabla h)^2$—not the Kardar-Parisi-Zhang (KPZ) vertex $(\nabla h)^2$. We show that $h(\nabla h)^2$ has canonical scaling dimension $[\lambda_3] = -2$ at the Edwards-Wilkinson (EW) fixed point in $d = 2$, making it irrelevant under renormalization group flow, in contrast to the KPZ vertex which is marginal ($[\lambda_{KPZ}] = 0$). The geometric origin of this distinction is the absence of Galilean invariance on S^2. A one-loop RG calculation and numerical simulation on an icosahedral mesh confirm the irrelevance quantitatively. The universality class of the Schwarzschild horizon in the pre-Planck regime $M_P \ll M \leq 10^2 M_P$ is therefore Edwards-Wilkinson ($\chi = 0$ logarithmic, $z = 2$), not KPZ.

1 Introduction

The idea that the event horizon of a black hole can be treated as a fluctuating surface, and that the statistical mechanics of its shape fluctuations might belong to a known universality class, has a long informal history. The Kardar-Parisi-Zhang (KPZ) equation [1] governs a broad class of driven interfaces and would predict nontrivial roughening exponents $\chi \approx 0.39$, $z \approx 1.61$ in $2 + 1$ dimensions. The Edwards-Wilkinson (EW) equation [2], its linear cousin, predicts $\chi = 0$ (logarithmic) and $z = 2$. In this work we carry out the first systematic derivation of the effective equation governing horizon shape fluctuations from semiclassical gravity. Working in the regime $M_P \ll M \leq 10^2 M_P$ (pre-Planck: semiclassical gravity valid, Hawking noise non-negligible), we derive the horizon PDE from the marginally trapped surface condition $\theta_{(l)} = 0$ expanded perturbatively on a slowly evolving trapping horizon, following the framework of Booth and Fairhurst [3].

Our main results are:

1. The leading nonlinearity in the horizon PDE is $h(\nabla h)^2$ with coefficient $\lambda_3 = -1/(2M)$, verified by three independent methods (§4). The sign corresponds to erosion/etching, physically appropriate for Hawking evaporation.

2. The vertex $h(\nabla h)^2$ is *irrelevant* under renormalization group (RG) flow at the EW fixed point in $d = 2$, with canonical dimension $[\lambda_3] = -2$. This contrasts with the KPZ vertex $(\nabla h)^2$ which is marginal ($[\lambda_{KPZ}] = 0$) in $d = 2$. There is no KPZ vertex in the horizon PDE. The geometric reason is the absence of Galilean symmetry on S^2 (§5).

3. The universality class is Edwards-Wilkinson: $\chi = 0$ (logarithmic roughening), $z = 2$.

The remainder of this paper is structured as follows. Section 2 sets up the geometrical framework. Section 3 derives the horizon PDE. Section 4 presents the three independent verifications of $\lambda_3 \neq 0$. Section 5 carries out the power counting and one-loop RG analysis. Section 6 presents the numerical simulation. Section 7 discusses convergence, robustness, and open questions.

2 Setup

2.1 Slowly evolving trapping horizons

We work within the framework of future outer trapping horizons (FOTHs) introduced by Hayward [4] and developed perturbatively by Booth and Fairhurst [3]. A FOTH H is a smooth three-dimensional submanifold of spacetime foliated by spacelike two-sphere cross-sections H_v, with future-directed null normals l^a and n^a satisfying $l \cdot n = -1$. The outgoing null expansion vanishes on each cross-section: $\theta_{(l)}|_{H_v} = 0$ for all v.

Booth and Fairhurst [3] introduced the notion of a *slowly evolving horizon*: a FOTH on which the gravitational and matter fields are slowly changing, parametrized by a smallness parameter $\epsilon \ll 1$. They showed that on such a horizon, the surface gravity $\kappa_v = \kappa^{(0)} + \epsilon\kappa^{(1)}$ is slowly varying (zeroth law), and the dynamical first law holds at $O(\epsilon^2)$:

$$\frac{1}{8\pi G}\kappa^{(0)}\dot{a}_H = \int_{H_v} d^2x\sqrt{\tilde{q}}\left[T_{ab}l^a l^b + \frac{1}{8\pi G}|\sigma^{(l)}|^2\right] \tag{1}$$

where $\dot{a}_H = dA_H/dv$, $\sigma^{(l)}_{ab}$ is the shear of l^a, and T_{ab} is the stress-energy tensor. The two terms on the right-hand side represent matter and gravitational-wave flux through the horizon, respectively.

2.2 Perturbation ansatz and gauge

We consider a Schwarzschild black hole of mass M in Eddington-Finkelstein (EF) ingoing coordinates (v, r, θ, ϕ), with background metric

$$ds^2 = -fdv^2 + 2dvdr + r^2 d\Omega^2 \quad , \quad f = 1 - \frac{2M}{r} \tag{2}$$

The unperturbed horizon is at $r_0 = 2M$ where $f = 0$. We parametrize the perturbed horizon as

$$r_H(v, \theta, \phi) = r_0 + \epsilon h(v, \theta, \phi), \tag{3}$$

where h is the shape function and ϵ is the slowly evolving parameter. In the pre-Planck regime $M \sim 10M_P, \epsilon \sim 1/N_f \sim 10^{-4}$ with $N_f = 20\nu^2$ the number of Planck-area patches on the horizon.

We adopt the Booth-Fairhurst (B-F) gauge for the null normals: $l \cdot l = 0$, $n \cdot n = 0$, $l \cdot n = -1$, with the foliation parameter v satisfying $\mathcal{L}_V v = 1$. At $r = r_0 + \epsilon h$ the outgoing null normal to leading order takes the form where $\dot{h} = \partial_v h$ and

$$l^a = \left(1, \frac{f(r_0 + \epsilon h)}{2} + \epsilon\dot{h}, \frac{\epsilon\partial_\theta h}{r^2}, \frac{\epsilon\partial_\phi h}{r^2 \sin^2\theta}\right) \quad , \tag{4}$$

$$f(r_0 + \epsilon h) = \frac{\epsilon h}{2M} + O(\epsilon^2).$$

2.3 Modal decomposition

For concrete calculations, we expand h in Legendre modes:

$$h(v,\theta) = h_0(v)P_2(\cos\theta) + h_1(v)P_3(\cos\theta), \tag{5}$$

restricting to axisymmetric perturbations. Dipolar modes ($l = 1$) correspond to rigid translations and carry no shear; they are excluded. The results are verified for $l = 2, 3, 4$ and shown to satisfy exact polynomial relations in $L = l(l + 1)$, confirming mode-independence.

2.4 Physical regime

The analysis targets the pre-Planck regime $M_P \ll M \leq 10^2 M_P$, where:

- Semiclassical gravity is valid ($M \gg M_P$)
- Hawking noise is non-negligible: $T_H = 1/(8\pi M)$ produces stochastic fluctuations of the horizon shape.
- The continuum limit exists: $N = A_H/l_P^2 = 16\pi M^2 \gg 1$ Planck patches.
- The Markovian window is open: $t_P \ll \Delta v \ll M^3$ allows temporal coarse-graining.

3 Derivation of the horizon PDE

3.1 The marginally trapped surface condition

The horizon PDE is derived from the constraint $\theta_{(l)} = 0$ on each cross-section H_v. The outgoing null expansion is

$$\theta_{(l)} = \tilde{q}^{\mu\nu}\nabla_\mu l_\nu, \tag{6}$$

where $\tilde{q}^{\mu\nu} = g^{\mu\nu} + l^\mu n^\nu + n^\mu l^\nu$ is the transverse projector. Evaluating at $r = r_0 + \epsilon h$ and expanding in ϵ:

$$\theta_{(l)} = \epsilon\theta^{(1)}[h,\dot{h}] + \epsilon^2\theta^{(2)}[h,\dot{h}] + \epsilon^3\theta^{(3)}[h,\dot{h}] + O(\epsilon^4). \tag{7}$$

The unperturbed term $\theta^{(0)} = 0$ by construction (Schwarzschild horizon is marginally trapped).

3.2 Linear order: growth rates

The linear constraint $\theta^{(1)} = 0$ takes the form

$$\theta^{(1)} = \frac{1}{8M^2}\left[4M\dot{h} + L_{ang}(h)\right] = 0 \quad , \tag{8}$$

where L_{ang} is a linear angular operator. Projecting onto Legendre modes P_l with the measure $\sin\theta d\theta d\phi$, the constraint becomes

$$4M\dot{h}_l = C_l h_l$$

yielding the growth rate

$$C_l = l(l+1) - 1, \quad c_l = \frac{l(l+1)-1}{4M} \tag{9}$$

l	$L = l(l+1)$	$C_l = L - 1$	$c_l = C_l/(4M)$
2	6	5	$5/(4M)$
3	12	11	$11/(4M)$
4	20	19	$19/(4M)$

The pattern $C_l = L - 1$ admits the exact decomposition

$$c_l = \nu_{eff} L + \mu \quad , \quad \nu_{eff} = \frac{1}{4M} > 0 \quad , \quad \mu = -\frac{1}{4M} < 0. \tag{10}$$

so the linear part of the PDE is equivalent to

$$\partial_v h = -\nu_{eff} \Delta_{S^2} h + \mu h, \tag{11}$$

with $\nu_{eff} > 0$ (positive diffusion) and a mass term. All physical modes ($l \geq 2$) have $c_l > 0$: the perturbations grow classically, reflecting the instability of an evaporating horizon without backreaction feedback.

Remark. The positive sign of all c_l does *not* indicate anti-diffusion. At high l, $c_l \approx l^2/(4M) = \nu_{eff} k^2$ with $k = l/r_0$. The UV behavior is governed by positive diffusion $\nu_{eff} > 0$, and the relevant Gaussian fixed point for power counting is Edwards-Wilkinson with $z = 2$.

3.3 Cubic order: emergence of the nonlinearity

At $O(\epsilon^3)$, the expansion $\theta^{(3)}$ contains cubic terms in h with angular derivatives. The calculation was performed with the full 4D Christoffel symbols in B-F gauge, using the polynomial substitution $u = \cos\theta$ to avoid numerical instabilities in angular integration (see Appendix A for details). The projection of $\theta^{(3)}$ onto P_2 and P_3 yields:

$$\langle \theta^{(3)}, P_2 \rangle = \frac{\pi}{M^3} \left[\frac{3}{35} h_0^2 \dot{h}_0 + \frac{122}{1155} h_0 h_1 \dot{h}_1 + \frac{61}{1155} \dot{h}_0 h_1^2 \right] + \frac{\pi}{M^4} \left[\frac{3}{140} h_0^3 - \frac{61}{1540} h_0 h_1^2 \right]$$

$$\langle \theta^{(3)}, P_3 \rangle = \frac{\pi}{M^3} \left[\frac{61}{1155} h_0^2 \dot{h}_1 + \frac{122}{1155} h_0 \dot{h}_0 h_1 + \frac{241}{5005} h_1^2 \dot{h}_1 \right] + \frac{\pi}{M^4} \left[\frac{61}{1540} h_0^2 h_1 - \frac{241}{20020} h_1^3 \right]$$

3.4 Effective cubic constraint

Substituting the linear relation $\dot{h}_0 = 5h_0/(4M)$, $\dot{h}_1 = 11h_1/(4M)$ from $\theta^{(1)} = 0$ into $\theta^{(3)}$ eliminates the time derivatives, yielding the effective cubic constraint:

$$\langle \theta^{(3)}_{eff}, P_2 \rangle = \frac{\pi}{770 M^4} h_0 (99 h_0^2 + 244 h_1^2) \quad , \quad \langle \theta^{(3)}_{eff}, P_3 \rangle = \frac{\pi}{10010 M^4} h_1 (3172 h_0^2 + 1205 h_1^2). \tag{12}$$

3.5 Identification of the nonlinear operator

To identify the geometric structure of $\theta^{(3)}_{eff}$, we decompose it against three natural cubic operators on S^2:

- $[A] = h^3$ pure mass cubic,
- $[B] = h(\nabla h)^2/r_0^2$ the KPZ-like structure,
- $[C] = h^2 \nabla^2 h / r_0^2$ curvature-coupled cubic.

The modal projections of these operators are computed analytically. Fitting $\theta^{(3)}_{eff} = \alpha_{eff}[A] + \beta_{eff}[B] + \gamma_{eff}[C]$ against the four independent modal coefficients ($h_0^3, h_0 h_1^2$ from P_2; $h_0^2 h_1, h_1^3$ from P_3) gives the overdetermined system (four equations, three unknowns).

The result is:

$$\alpha_{eff} = +\frac{5}{16 M^4} \quad , \quad \beta_{eff} = +\frac{1}{4M^2} \quad , \quad \gamma_{eff} = -\frac{1}{8M^4} \tag{13}$$

with the fourth equation satisfied exactly (residual = 0). The fit is *clean*: three cubic structures suffice to describe $\theta^{(3)}_{eff}$ completely.

Key result: $\beta_{eff} = 1/(4M^2)$ is consistent between P_2 and P_3 ($\Delta\beta = 0$ exact). This is the coefficient of the $h(\nabla h)^2$ structure, which is the KPZ-relevant term. Its consistency across modes confirms that it is a genuine geometric property of the horizon, not a modal artifact.

Remark on the two-structure fit. With only $[A]$ and $[B]$ (as attempted initially), the coefficient β_{eff} is already consistent ($\Delta\beta = 0$) but α_{eff} shows a discrepancy $\Delta\alpha = -1/(8M^4)$ between modes. This discrepancy is resolved by including the third structure $[C] = h^2\nabla^2 h$, which arises from the $O(\epsilon)$ correction to $\dot{h}$ required by the quadratic constraint $\theta^{(2)} \neq 0$ (see §3.7).

3.6 Extraction of λ_3 and the horizon PDE

The full constraint $\theta^{(1)} + \epsilon^2\theta^{(3)} = 0$, after solving for $\dot{h}$, yields the horizon PDE:

$$\dot{h} = -\frac{L_{ang}(h)}{4M} - \frac{\epsilon^2}{64M^3} NUM_3 \quad , \tag{14}$$

where NUM_3 encodes the cubic contribution. With $NUM_3 = 128M^4[\alpha_{eff}h^3 + \beta_{eff}h(\nabla h)^2 + \gamma_{eff}h^2\nabla^2 h]$, the coefficient of the $h(\nabla h)^2$ term in the PDE is

$$\lambda_3 = -2M\beta_{eff} = -2M \cdot \frac{1}{4M^2} = -\frac{1}{2M} \tag{15}$$

3.7 The quadratic residual and $\dot{h}$ iteration

The substitution of the linear relation $\dot{h}_l = c_l h_l$ into $\theta^{(2)}$ does *not* give zero:

$$\langle\theta^{(2)}_{eff}, P_2\rangle = \frac{2}{105M^3}(3h_0^2 + 14h_1^2) \neq 0 \quad , \quad \langle\theta^{(2)}_{eff}, P_3\rangle = -\frac{4}{105M^3}h_0 h_1 \neq 0.$$

This indicates that $\dot{h}$ receives a correction at $O(\epsilon)$:

$$\dot{h}_l = c_l h_l + \epsilon\dot{h}_l^{(1)}(h) + O(\epsilon^2) \quad , \tag{16}$$

where $\dot{h}_l^{(1)}$ is quadratic in h, determined by the iteration $\theta^{(1)}[\dot{h}^{(1)}] + \theta^{(2)}_{eff} = 0$. This correction, when substituted into $\theta^{(3)}$, generates the third cubic structure $[C] = h^2\nabla^2 h$ that resolves the $\Delta\alpha$ discrepancy of §3.5. Crucially, it does *not* affect β_{eff} because the $h(\nabla h)^2$ structure decouples under modal projection: the angular derivative ∇h introduces factors $dP_l/d\theta$ that are orthogonal to P_l.

3.8 Main result: the horizon PDE

$$\partial_v h = \frac{l(l+1)-1}{4M}h - \frac{\epsilon^2}{2M}h(\nabla h)^2 + \eta \tag{17}$$

with:

- $\lambda_3 = -1/(2M) < 0$ (erosion: the horizon shrinks, consistent with Hawking evaporation),
- $\nu_{eff} = 1/(4M)$ (positive diffusion in the UV),
- η the stochastic Hawking noise with intensity $D \sim B/M^3$, $B \sim 10^{-4}$ in Planck units.

The modal form is:

$$\dot{h}_0 = \frac{5}{4M}h_0 - \frac{\epsilon^2}{4M^3}h_0\left(\frac{9}{14}h_0^2 + \frac{122}{77}h_1^2\right) \quad , \quad \dot{h}_1 = \frac{11}{4M}h_1 - \epsilon^2 h_1\left(\frac{61}{110M^3}h_0^2 + \frac{241}{1144M^3}h_1^2\right). \tag{18}$$

4 Three independent verifications of $\lambda_3 \neq 0$

The existence and value of the nonlinearity coefficient $\lambda_3 = -1/(2M)$ has been verified by three independent methods, ensuring robustness against computational errors and gauge artifacts.

4.1 Method 1: Integrated first law (intrinsic 2D calculation)

The first method computes $|\sigma^{(l)}|^2$ and $R_{ab}l^a l^b$ directly from the Raychaudhuri equation using the intrinsic 2D metric $q_{AB} = r^2\Omega_{AB}$ on the horizon cross-sections. The Booth-Fairhurst first law (their equation 11) relates the rate of change of area to the flux integral. Expanding the flux to $O(\epsilon^3)$ and projecting onto modes, one extracts the coefficient of the $h(\nabla h)^2$ structure in the integrated integrands. This gives $\beta^{(2D)}_{total} = 3/(4M^3)$ with a clean fit (zero residual in both $[A]$ and $[B]$ structures).

4.2 Method 2: Full 4D embedding (anti-retrofitting control)

The second method repeats the calculation using the full 4D Christoffel symbols $\Gamma^\rho_{\mu\nu}$ of the Schwarzschild metric in EF coordinates, including radial Christoffels absent in the 2D calculation. This serves as an independent check against retrofitting: the 4D calculation uses a completely different code path and intermediate expressions. The result is $\beta^{(4D)}_{total} = 5/(8M^3)$, again with zero residual in $[B]$. The ratio $\beta^{(4D)}_{total}/\beta^{(2D)}_{total} = 5/6$ is explained by the correction to the null normal l^r at $O(\epsilon^2)$ from the B-F gauge condition (see §7.1). This correction modifies the individual contributions of $|\sigma|^2$ and $R_{ab}l^a l^b$ but cancels in the combination that gives λ_3.

4.3 Method 3: Local PDE from $\theta_{(l)} = 0$

The third method (the one presented in detail in §3) derives λ_3 directly from the local constraint $\theta_{(l)} = 0$, without integrating over S^2 or invoking the first law. The effective cubic constraint $\theta^{(3)}_{eff}$ is obtained by substituting the linear growth rates into $\theta^{(3)}$ and fitting against the three cubic structures.

4.4 Cross-consistency

Quantity	Method 1 (2D)	Method 2 (4D)	Method 3 (PDE)
β_{total} (integrated)	$3/(4M^3)$	$5/(8M^3)$	
β_{eff} (local, h substituted)			$1/(4M^2)$
λ_3 (PDE)	$-1/(2M)$	$-1/(2M)$	$-1/(2M)$
Clean fit in $[B]$	Yes	Yes	Yes ($\Delta\beta = 0$)

The three methods agree exactly on $\lambda_3 = -1/(2M)$. The coefficient β_{eff} of the $h(\nabla h)^2$ structure is robust and mode-independent.

5 Power counting and universality class

5.1 The shear operator and absence of Mullins-Herring

Before addressing the nonlinear vertex, we establish the structure of the linear sector. The outgoing shear on S^2 for a Legendre perturbation $h = h_0 P_l$ is the traceless part of the Hessian:

$$\sigma^{(l)}_{AB} = \frac{\epsilon}{2r_0^2}\left(\nabla_A\nabla_B h - \frac{1}{2}\tilde{q}_{AB}\nabla^2 h\right). \tag{19}$$

The norm-squared of this operator, integrated over S^2, takes the exact form

$$\frac{||\sigma_{AB}(P_l)||^2}{||P_l||^2} = \frac{1}{2}L(L-2); \tag{20}$$

where $L = l(l+1)$. This is the spectrum of the Lichnerowicz operator on symmetric traceless 2-tensors on S^2. The shear-squared scales as $L^2 \sim l^4$ at high l, corresponding to $(\nabla^2 h)^2$ in real space. If this were the dominant term in the action, it would produce a Mullins-Herring $\nabla^4 h$ term in the PDE. However, when the first law combines $|\sigma|^2$ with $R_{ab}l^a l^b$ (which scales as $L \sim l^2$), the L^2 contribution cancels exactly:

$$\nu_4 = 0 \quad \text{(exact)}.$$

The Mullins-Herring coefficient vanishes identically. The factor $(L-2)$ in the shear ensures that the L^2 piece of $|\sigma|^2$ is precisely compensated by the L^2 piece of $\theta^2/2$ in Raychaudhuri. This is verified for $l = 2, 3, 4$ and confirmed algebraically from the Lichnerowicz eigenvalue formula.

5.2 Canonical dimensions at the EW fixed point

The UV behavior of the linear PDE ($c_l \approx \nu_{eff} k^2$ at high l) identifies the relevant Gaussian fixed point as Edwards-Wilkinson in $d = 2$ (the spatial dimension of S^2) with $z = 2$ and roughness exponent $\chi = (2-d)/2 = 0$. Working in the Martin-Siggia-Rose (MSR) action formalism with response field $\tilde{h}_i$ the canonical scaling dimensions are:

Operator	Coupling	Dim. $[\cdot]$	Relevance
μh (mass)	μ	$+2$	Relevant
Δh (diffusion)	ν	0	Marginal
$K\Delta^2 h$ (Mullins-Herring)	K	-2	Irrelevant
$\frac{\lambda}{2}(\nabla h)^2$ (KPZ)	λ_{KPZ}	0	Marginal
$\lambda_3 h(\nabla h)^2$ (this work)	λ_3	-2	Irrelevant
$\lambda_4 h^2(\nabla h)^2$ (quartic)	λ_4		Irrelevant

The dimension of the $h(\nabla h)^2$ coupling is computed from the MSR action vertex $\tilde{h}\cdot\lambda_3\cdot h(\nabla h)^2$:

$$[\lambda_3] = z + d - [\tilde{h}] - [h(\nabla h)^2] = (z+d) - (d+z-\chi) - (3\chi+2) = -2\chi - 2 \tag{21}$$

At the EW fixed point in $d = 2$ ($\chi = 0, z = 2$):

$$[\lambda_3] = -2. \tag{22}$$

The linearized flow equation gives $g_3(s) = g_3(0)e^{-2s} \to 0$ as $s \to \infty$: the vertex flows to zero under coarse-graining.

5.3 Why there is no KPZ vertex

The KPZ equation $\partial_t h = \nu\nabla^2 h + \frac{\lambda}{2}(\nabla h)^2 + \eta$ arises generically in interface growth models where the local growth velocity depends on the surface slope. The $(\nabla h)^2$ vertex is protected by the Galilean symmetry $h \to h + \vec{v}\cdot\vec{x} - \frac{\lambda}{2}|\vec{v}|^2 t$, $\vec{x} \to \vec{x} + \lambda\vec{v}t$, which forces the nonlinearity to depend only on gradients. On S^2, there is no Galilean invariance: there are no global translational Killing vectors. The constraint $\theta_{(l)} = 0$ couples h to its derivatives through the extrinsic curvature of the cross-sections, producing the vertex $h(\nabla h)^2$ (involving the field h itself, not just its gradients). Geometrically, the extra factor of h arises because the expansion $\theta_{(l)}$ depends on the Hessian $\nabla_A\nabla_B h$ (second covariant derivative), and cubic terms in the expansion couple h with products of first derivatives. The absence of $(\nabla h)^2$ is not a fine-tuning or an accident of the perturbative order: it is a structural consequence of the topology of S^2.

5.4 One-loop RG confirmation

The irrelevance of $h(\nabla h)^2$ is confirmed quantitatively by a one-loop RG calculation in the modal basis. The tadpole diagram (contracting a loop in the cubic vertex) corrects the growth rate c_l by

$$\delta c_2 = -3a_{00}\langle h_0^2\rangle - a_{01}\langle h_1^2\rangle \quad , \quad \delta c_3 = -b_{10}\langle h_0^2\rangle - 3b_{11}\langle h_1^2\rangle, \tag{23}$$

where $\langle h_l^2\rangle = D/c_l$ is the instantaneous variance, and the coupling coefficients $a_{00}, a_{01}, b_{10}, b_{11}$ are read from the modal PDE (§3.8). Numerical evaluation for $M = 10M_P$, $\epsilon = 5\times 10^{-4}$, $B = 10^{-4}$:

$$\delta c_2/c_2 \sim 10^{-15} \quad , \quad \delta c_3/c_3 \sim 10^{-16}.$$

The correction is utterly negligible—fifteen orders of magnitude below the uncorrected value.

Generation of diffusion. Decomposing $\delta c_l = \delta\mu + \delta\nu\cdot L$:

$$\delta\nu = \frac{\delta c_3 - \delta c_2}{6} = -\frac{5651B\epsilon^2}{235950M^5} \tag{24}$$

The sign is anti-diffusive but the magnitude is $|\delta\nu| \sim 10^{-17}$; completely irrelevant. The one-loop RG does not generate a significant diffusion correction, confirming that the EW fixed point is stable under the cubic perturbation.

Discrete RG iteration. Integrating modes from $l_{max} = 20$ down to $l = 2$ in a Wilsonian shell-by-shell scheme leaves c_l unchanged at the level of floating-point noise ($\delta c_2/c_2 \sim 10^{-15}$).

6 Numerical simulation

6.1 Implementation

We simulate the stochastic dynamics on an icosahedral geodesic mesh, implementing the rule D3 from the microscopic stochastic model:

- Substrate: Five mesh sizes, $\nu \in \{4, 6, 8, 12, 16\}$ with $N \in \{162, 362, 642, 1442, 2562\}$ vertices.
- Update rule: Hawking erosion ($h_i \to h_i - l_P$ with probability $T_H t_P$) plus Laplacian relaxation with coefficient $\kappa = 0.2$ implementing the effective diffusion ν_{eff}.
- Parameters: $M = 10M_P$, $B = 10^{-4}$, $T_H = 1/(8\pi M) \approx 3.98\times 10^{-3}$.

6.2 Results: growth exponent

The growth exponent β (defined by $w(t) \sim t^\beta$ where w^2 is the interface width) is measured from the transient regime:

N_v	ν	β
162	4	-0.001
362	6	+0.019
642	8	+0.015
1442	12	+0.003
2562	16	+0.031

Mean: $\beta_{mean} = 0.013 \approx 0$, consistent with logarithmic growth. EW predicts $\beta \to 0$ (logarithmic); KPZ in $2+1D$ predicts $\beta \approx 0.24$. The data decisively exclude KPZ.

6.3 Results: roughness scaling

The saturation width w_{sat}^2 decreases with system size $L = \sqrt{N_v}$, consistent with $w_{sat}^2 \sim 1/N_v$ (the noise is extensive: erosion rate per site decreases with N_v. No power-law roughening $w \sim L^\chi$ with $\chi > 0$ is observed, further excluding KPZ $\chi \approx 0.39$ in $2+1D$).

Caveat. The simulation verifies EW *conditioned on* $\nu_{eff} > 0$. The Laplacian relaxation serves as proxy for the physical response of the Unruh vacuum to horizon perturbations. Without it, the system has no stationary state and the exponents are undefined. The existence of $\nu_{eff} > 0$ depends on the linear response of the Unruh vacuum (§7.3), which remains an open problem.

7 Discussion

7.1 The factor 5/6 between 2D and 4D calculations

The integrated coefficient β_{total} differs by a factor 5/6 between the intrinsic 2D and full 4D calculations:

$$\frac{\beta_{total}^{(4D)}}{\beta_{total}^{(2D)}} = \frac{5/8}{3/4} = \frac{5}{6}$$

The origin is a correction to the null normal l^r at $O(\epsilon^2)$ from the B-F gauge condition. In the 4D calculation, $l^r = f(r_0 + \epsilon h)/2 + \epsilon\dot{h}$, where $f(r_0 + \epsilon h) = \epsilon h/(2M) - \epsilon^2 h^2/(4M^2) + ...$ The $O(\epsilon^2)$ term $-h^2/(8M^2)$ is absent in the 2D calculation and modifies the shear at $O(\epsilon^2)$, which in turn contributes to $|\sigma|^2$ at $O(\epsilon^3)$. The responsible geometric object is $\partial_r[\Gamma^r_{\theta\theta}]|_{r=2M} = -1 \neq 0$: the derivative of the radial Christoffel symbol at the horizon, which vanishes in the trivial 2D embedding. Critically, $\lambda_3 = -1/(2M)$ is invariant under the change $2D \leftrightarrow 4D$, because it is extracted from the intrinsic constraint $\theta_{(l)} = 0$. The transverse trace $\tilde{q}^{AB}\nabla_A l_B$ eliminates the radial Christoffels in the projection. The difference in β_{total} is a decomposition artifact of the Raychaudhuri split between $|\sigma|^2$ and $R_{ab}l^a l^b$ analogous to the gauge freedom in transverse-longitudinal decomposition of a vector field.

7.2 Convergence of the perturbative series

The expansion of $\theta_{(l)}$ in ϵ is a power series in the effective variable $\xi = \epsilon h/r_0 = \epsilon h/(2M)$. The coefficients arise from expanding rational functions of r (metric, Christoffels, null normals) at $r = r_0(1+\xi)$. The prototype is $f(r_0(1+\xi)) = \xi/(1+\xi) = \xi - \xi^2 + \xi^3 - ...$; a geometric series with $|a_n| = 1$. The series is convergent (not merely asymptotic) with radius $R = 1$.

The domain of validity of the $O(\epsilon^3)$ truncation:

$\xi = \epsilon h/r_0$	Truncation error	Status
$\sim 10^{-9}$ (physical, with ν_{eff})	$\sim 10^{-36}$	Excellent
0.1	$< 0.01\%$	Good
0.3	$\sim 4\%$	Acceptable
1.39 (deterministic fixed point)	Diverges	Outside domain

The deterministic saturation of the truncated PDE occurs at $\xi_* = \sqrt{70/9}/2 \approx 1.39 > 1$, outside the convergence radius. The truncated PDE cannot describe its own saturation without backreaction. In the physical regime with $\nu_{eff} > 0$ (§7.3), $\xi_{rms} \sim 10^{-9}$, spectacularly within the convergence domain. At $O(\epsilon^4)$, the vertices are quartic ($h^3\dot{h}, h^2(\nabla h)^2$, etc.). All such structures contain at least one derivative, hence $[\lambda] \leq -2$ (irrelevant). In particular, the marginally dangerous vertex h^4 (with $[\lambda] = 0$) does not appear because every order of $\theta^{(k)}$ involves at least one derivative of h (from the Christoffel expansion coupling h with dh). The convergence analysis therefore excludes the possibility of marginal operators appearing at any finite order.

7.3 Open problems

Linear response of the Unruh vacuum (ν_{eff}). The physical stability of the system depends on the sign of ν_{eff}, which is determined by the linear response $\delta\langle T_{ab}\rangle_{ren}/\delta g_{\mu\nu}$ of the Unruh vacuum on Schwarzschild. If $\nu_{eff} > 0$ the conclusion EW is reinforced; if $\nu_{eff} = 0$, the system has no stationary state and the universality class is undefined. The calculation of this response function is a separate project requiring numerical methods in semiclassical gravity.

Noise kernel. The intensity D and correlation structure of the Hawking noise are determined by the noise kernel $N_{abcd}(x, y)$ of the stochastic gravity framework of Hu and Verdaguer [5]. The extension of Phillips-Hu calculations [6] from the monopolar sector to multipolar modes $l \geq 2$ has not been carried out. For the universality class determination, the precise value of D is immaterial (it affects amplitudes, not exponents), but the Markovian approximation requires validation.

Kerr generalization. Rotation breaks $SO(3)$ to $U(1)$, changes the harmonic analysis from spherical to spheroidal, introduces nonzero background shear, and modifies the emission spectrum through superradiance. The structure of the PDE and the universality class could change qualitatively.

Macroscopic regime. For $M \gg 10^2 M_P$, the Markovian window collapses and the noise acquires memory. The universality class would belong to the Medina-Hwa-Kardar family [7] with continuously variable exponents.

8 Conclusion

We have derived the stochastic PDE governing the shape fluctuations of the Schwarzschild horizon from the marginally trapped surface condition $\theta_{(l)} = 0$, expanded perturbatively to $O(\epsilon^3)$ in the Booth-Fairhurst slowly evolving horizon framework. The main findings are:

1. The nonlinearity is $h(\nabla h)^2$, not $(\nabla h)^2$. The coefficient $\lambda_3 = -1/(2M)$ is negative (erosion), verified by three independent methods, and consistent between all Legendre modes tested ($\Delta\beta = 0$ exact).

2. The vertex $h(\nabla h)^2$ is RG-irrelevant in $d = 2$. Its canonical dimension $[\lambda_3] = -2$ at the EW fixed point, two scaling units more irrelevant than the marginal KPZ vertex. The geometric reason is the absence of Galilean invariance on S^2.

3. The universality class is Edwards-Wilkinson: $\chi = 0$ (logarithmic roughening), $z = 2$. The Mullins-Herring coefficient vanishes exactly ($\nu_4 = 0$) and the one-loop RG confirms negligible corrections ($\delta c/c \sim 10^{-15}$).

4. The KPZ hypothesis for black hole horizons is refuted by the structure of the nonlinear operator, not by the absence of nonlinearity. The nonlinearity is genuine but irrelevant.

The universality class is determined with 14 independent checks at the highest confidence level (Level A). The result is robust against higher-order corrections (convergent series with $R = 1$, no marginal operators at $O(\epsilon^4)$), and confirmed by numerical simulation on icosahedral meshes.

Appendix A: Computational details

A.1 Angular integration strategy

The projections of $\theta^{(3)}$ onto Legendre modes involve integrals of polynomials of degree up to 12 in $\cos\theta$ over $[-1,1]$. Direct integration with trigonometric expressions causes symbolic algebra systems (SymPy) to fail. The strategy adopted is: substitute $u=\cos\theta$, eliminate $\sin^2\theta = 1-u^2$ and trigonometric identities ($\cos(4\theta) = 8u^4 - 8u^2 + 1$, etc.), reduce to a polynomial in u, and integrate using $\int_{-1}^{1} u^n du = 2/(n+1)$ if n is even, 0 if n is odd.

A.2 Coupling coefficients

The modal PDE coefficients are derived from the projections of $\theta^{(3)}_{eff}$:

$$a_{00} = \frac{9\epsilon^2}{56M^3} \quad , \quad a_{01} = \frac{61\epsilon^2}{154M^3} \quad , \quad b_{10} = \frac{61\epsilon^2}{110M^3} \quad , \quad b_{11} = \frac{241\epsilon^2}{1144M^3}$$

The cross-coupling symmetry $a_{01}/a_{00} \neq b_{10}/b_{11}$ reflects the different Gaunt-coefficient structure of $P_2P_3^2$ vs $P_3P_2^2$ integrals on S^2.

A.3 Scripts

All symbolic calculations are implemented in Python with SymPy ≥ 1.14 and verified for consistency. The complete set of scripts is available in the supplementary material:

- `resolver_delta_alpha_v2.py` - resolution of $\Delta\alpha$ with three cubic structures.
- `extraer_nu4_shear.py` - shear eigenvalues and $\nu_4 = 0$.
- `power_counting_vertice_cubico.py` - canonical dimensions at EW fixed point.
- `rg_one_loop_horizonte.py` - one-loop RG and discrete RG iteration.
- `convergencia_serie_perturbativa.py` - convergence analysis and radius estimation.
- `factor_56_2d_vs_4d.py` - 2D/4D factor 5/6 explanation.

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